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Spectral geometry, homogeneous spaces and differential forms with finite Fourier series

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Abstract
Let G be a compact Lie group acting transitively on Riemannian manifolds M_i and let $\pi : M_1 \to M_2$ be a G-equivariant Riemannian submersion. We show that a smooth differential form ϕ on M_2 has finite Fourier series on M_2 if and only if the pull back $\pi^*\phi$ has finite Fourier series on M_1.

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1. Introduction

The spectral geometry of homogeneous Riemannian submersions has been discussed by many authors in a variety of physical contexts. For example, Bérard-Bergery, and Bourguignon [4] study the Laplacian of a Riemannian submersion and provide an application to quantum physics. Also, Boiteux [5] studies the Coulumb potential via a fiber bundle formulation of mechanics, and we direct the reader who is interested in this particular physical application to remark 1.3; see [13, 17] for subsequent work. The spectral geometry of homogeneous Riemannian submersions also plays an important role in the study of non-bijective canonical transformations; we refer, for example, to the discussion in Lambert and Kibler [14] (see also [6, 11, 15, 16] for later work). Gilkey, Leahy and Park [9] study the spectral geometry of the Hopf fibration $S^1 \to S^3 \to S^2$ and provide an overview of the potential physical applications. Their work continues in [10], where the authors give a more extensive discussion of these applications; see also Bao and Shen [1].

Let M be a compact smooth closed Riemannian manifold of dimension m, and let Δ_M^p be the Laplace–Beltrami operator acting on the space $C^\infty(\Lambda^p M)$ of smooth p-forms. Let $\mathrm{Spec}(\Delta_M^p)$ be the spectrum of Δ_M^p; this is a discrete countable set of non-negative real

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numbers. The associated eigenspaces $E\big(\lambda, \Delta_M^p\big)$ are finite dimensional and there is a complete orthonormal decomposition

$$L^2(\Lambda^p M) = \oplus_{\lambda\in\mathrm{Spec}(\Delta_M^p)} E\big(\lambda, \Delta_M^p\big) \tag{1a}$$

which we may use to decompose a smooth p-form ϕ on M in the form $\phi = \sum_\lambda \phi_\lambda$, where $\phi_\lambda \in E\big(\lambda, \Delta_M^p\big)$. We say ϕ has *finite Fourier series on M* if this is a finite sum. If $p = 0$ and if $M = S^1$, then this yields, modulo a slight change of notation, the classical Fourier series decomposition $f(\theta) = \sum_n a_n\,\mathrm{e}^{in\theta}$ and a function has a finite Fourier series on the circle if and only if it is a trigonometric polynomial. There is an extensive literature on the subject with appropriate physical applications, several representative items being [2, 3, 7, 8, 18].

We say that M is a *homogeneous space* if there is a compact Lie group G which acts transitively on M by isometries; if H is the isotropy subgroup associated with some point $P \in M$, then we may identify $M = G/H$. We may choose a left-invariant metric $\tilde{g}$ on G so g is the induced metric or, equivalently, that $\pi : (G, \tilde{g}) \to (M, g)$ is a Riemannian submersion. The following is the main result of this paper:

Theorem 1.1. *Let $\pi : G \to G/H$, where H is a Lie subgroup of a compact Lie group G. Let $\tilde{g}$ be a left-invariant Riemannian metric on G and let g be the induced Riemannian metric on G/H. Then a p-form ϕ on G/H has finite Fourier series on G/H if and only if $\pi^*\phi$ has finite Fourier series on G.*

There is an associated corollary which is useful in applications.

Corollary 1.2. *Let G be a compact Lie group acting transitively on Riemannian manifolds M_1 and M_2. Let $\pi : M_1 \to M_2$ be a G-equivariant Riemannian submersion. If ϕ is a smooth p-form on M_2, then ϕ has finite Fourier series on M_2 if and only if $\pi^*\phi$ has finite Fourier series on M_1.*

Remark 1.3. The Hopf fibration $\pi : S^{2n+1} \to \mathbb{CP}^n$ is a $U(n+1)$ equivariant Riemannian submersion which is an important non-canonical transformation used to study the Coulumb problem, see, for example, the discussion in [5]. Corollary 1.2 shows ϕ has finite Fourier series on $\mathbb{CP}^n$ if and only if $\pi^*\phi$ has finite Fourier series on S^{2n+1}.

2. The proof of theorem 1.1

The central ingredient in our discussion is the classical Peter–Weyl theorem [12]. Let $\mathrm{Irr}(G)$ be the collection of equivalence classes of irreducible finite dimensional representations of G; if $\rho \in \mathrm{Irr}(G)$, let V_ρ be the associated representation space. The Hilbert space structure on $L^2(\Lambda^p(G))$ depends on the particular Riemannian metric which is chosen. However different Riemannian metrics give rise to equivalent norms so the Banach space structure on this space is invariantly defined; this is a minor observation which will be useful in section 4. Left multiplication defines an action of G on $L^2(\Lambda^p(G))$. This action decomposes as a direct sum

$$L^2(\Lambda^p G) = \oplus_{\rho\in\mathrm{Irr}(G)} W_\rho, \tag{2a}$$

where each W_ρ is a finite dimensional invariant subspace of $L^2(\Lambda^p(G))$ which is isomorphic to the direct sum of finite number of copies of V_ρ. If Φ is a smooth p-form on G, we may use equation (2a) to decompose $\Phi = \sum_\rho \Phi_\rho$ for $\Phi_\rho \in W_\rho$. We say that Φ has *finite G-representation series* on G if this sum is finite; we emphasize that this notion is independent of the particular Riemannian metric chosen.

Since we have taken the induced metric on G/H, the map π is a Riemannian submersion. Thus we have a pointwise estimate $|\pi^*\phi(g)| = |\phi(\pi g)|$. Since the volume form on G is bi-invariant, the fibers have constant volume. Thus

$$|\pi^*\phi|_{L^2(\Lambda^p(G))} = \sqrt{\mathrm{vol}(H)}|\phi|_{L^2(\Lambda^p(G/H))}.$$

Consequently π^* is an injective G-equivariant map from $L^2(\Lambda^p(G/H))$ to $L^2(\Lambda^p(G))$ with closed image. The decomposition

$$L^2(\Lambda^p G) = \pi^*(L^2(\Lambda^p(G/H))) \oplus \{\pi^*(L^2(\Lambda^p(G/H)))\}^\perp$$

is G-equivariant. We therefore have an orthogonal direct sum decomposition of $L^2(\Lambda^p(G/H))$ as a representation space for G in the form

$$L^2(\Lambda^p(G/H)) = \oplus_{\rho\in\mathrm{Irr}(G)} X_\rho, \qquad (2b)$$

where

$$\pi^* X_\rho = W_\rho \cap \pi^*(L^2(\Lambda^p(G/H))). \qquad (2c)$$

We say that a p-form ϕ on G/H has *finite G-representation series* if the expansion $\phi = \sum_\rho \phi_\rho$ given by equation (2*a*) is finite. Theorem 1.1 will follow from the following:

Lemma 2.1. *Adopt the notation established above. Let ϕ be a p-form on G/H. Fix a left-invariant $\tilde{g}$ metric on G and let g be the induced metric on G/H. The following assertions are equivalent:*

(i) ϕ has finite Fourier series on G/H.
(ii) ϕ has finite G-representation series on G/H.
(iii) $\pi^*\phi$ has finite Fourier series on G.
(iv) $\pi^*\phi$ has finite G-representation series on G.

We remark that elliptic regularity shows such a ϕ is necessarily smooth.

Proof. The equivalence of assertions (ii) and (iv) is immediate from equation (2*a*). We argue as follows to prove that assertion (i) implies assertion (ii). Suppose that ϕ has finite Fourier series on G/H. Since G acts by isometries, G commutes with the Laplacian. Thus $E(\lambda, \Delta^p_{G/H})$ is a finite dimensional representation space for G. Only a finite number of representations occur in the representation decomposition of $E(\lambda, \Delta^p_{G/H})$ and thus any eigen p-form on G/H has finite G-representation series on G/H; more generally, of course, any finite sum of eigen p-forms on G/H has finite G-representation series on G/H. This shows that assertion (i) implies assertion (ii); a similar argument shows assertion (iii) implies assertion (iv).

Each representation appears with finite multiplicity in $L^2(\Lambda^p(G/H))$. Thus each representation appears in the decomposition of $E(\lambda, \Delta^p_{G/H})$ for only a finite number of λ. Thus any element of X_ρ has finite Fourier series and more generally any p-form on G/H with finite G-representation series has finite Fourier series. Thus assertion (ii) implies assertion (i); similarly, assertion (iv) implies assertion (iii). □

3. The proof of corollary 1.2

Let $\pi : M_1 \to M_2$ be a G-equivariant Riemannian submersion; this means that we may express $M_i = G/H_i$, where $H_1 \subset H_2 \subset G$. Let $\pi_i : G \to G/H_i$ be the natural projections. We then have $\pi\pi_1 = \pi_2$ and thus $\pi_2^* = \pi_1^*\pi^*$. Let ϕ be a smooth p-form on G/H_2. We apply theorem 1.1 to derive the following chain of equivalent statements from which corollary 1.2 will follow:

(i) ϕ has finite Fourier series on G/H_2.
(ii) $\pi_2^*\phi$ has finite Fourier series on G.
(iii) $\pi_1^*(\pi^*\phi)$ has finite Fourier series on G.
(iv) $\pi^*\phi$ has finite Fourier series on G/H_1.

4. Conclusions and open problems

Our methods in fact show a bit more. Let g_i be two left-invariant metrics on G and let ϕ be a smooth p-form on G. Then ϕ has finite Fourier series with respect to g_1 if and only if ϕ has finite Fourier series with respect to g_2 since both conditions are equivalent to ϕ having finite representation series and this notion is independent of the particular metric chosen.

Cayley multiplication defines a Riemannian submersion $\pi : S^7 \times S^7 \to S^7$. The group of isometries commuting with this action does not, however, act transitively on $S^7 \times S^7$ and theorem 1.1 is not applicable. Our research continues in this area as this example has important physical applications (see, for example, the discussion in Lambert and Kibler [14]).

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