Monatsh Math (2008)
DOI 10.1007/s00605-008-0534-4
Printed in The Netherlands

Monatshefte für
Mathematik

Remarks on η-Einstein unit tangent bundles

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Communicated by D. V. Alekseevsky

Received May 16, 2007; accepted in final form September 18, 2007

Abstract. We study the geometric properties of the base manifold for the unit tangent bundle satisfying the η-Einstein condition with the canonical contact metric structure. One of the main theorems is that the unit tangent bundle of 4-dimensional Einstein manifold, equipped with the canonical contact metric structure, is η-Einstein manifold if and only if the base manifold is the space of constant sectional curvature 1 or 2.

2000 Mathematics Subject Classification: 53C25, 53D10
Key words: Unit tangent bundle, η-Einstein manifold, contact metric structure

1. Introduction

We consider the η-Einstein condition, which is suitable for contact metric manifold in general, that is, the Ricci tensor is of the form $\rho(\overline{X},\overline{Y}) = \alpha\, g(\overline{X},\overline{Y}) + \beta\,\eta(\overline{X})\eta(\overline{Y})$ with α and β being smooth functions. In [4], Boeckx and Vanchecke determined the unit tangent bundles which are Einstein with respect to the canonical contact metric structure. In the present paper, we shall extend their result to the η-Einstein case. The scalar curvature of an η-Einstein contact metric manifold is not necessarily constant in general, however, for some special η-Einstein contact metric manifolds, we may expect the scalar curvature to be constant. The main theorems are the following:

Theorem 1. *Let M be an $n(\geqslant 2)$-dimensional Riemannian manifold and T_1M be the unit tangent bundle of M equipped with the canonical contact metric structure. If T_1M is an η-Einstein manifold, then α and β are both constant valued ones on T_1M.*

Let τ be the scalar curvature of M, ρ be the Ricci curvature tensor of M, R be the Riemann curvature tensor of M and $\bar{\tau}$ be the scalar curvature of T_1M. Then we have the following theorems.

Theorem 2. *Let $(T_1M, \eta, \bar{g}, \phi, \xi)$ be an η-Einstein manifold. Then τ, $|\rho|^2$, $|R|^2$, and $\bar{\tau}$ are all constant.*

Theorem 3. *Let M be a 4-dimensional Einstein manifold and $(T_1M, \eta, \bar{g}, \phi, \xi)$ be the unit tangent bundle of M equipped with the canonical contact metric structure. Then T_1M is an η-Einstein manifold if and only if (M, g) is the space of constant sectional curvature 1 or 2.*

Question 1. *Can we extend the above Theorem 3 to higher dimensional cases?*

From our arguments in the present paper, the following question will naturally arise:

Question 2. *Does there exist $n(\geqslant 4)$ dimensional Riemannian manifold which is not a space of constant sectional curvature 1 or $n-2$, whose unit tangent bundle is η-Einstein?*

In the last section, we consider η-Einstein unit tangent bundles of some special base Riemannian manifolds.

2. Unit tangent bundle with contact metric structure

First, we give some preliminaries on a contact metric manifold. We refer to [2] for more details. A differentiable $(2n-1)$-dimensional manifold $\overline{M}$ is said to be a *contact manifold* if it admits a global 1-form η such that $\eta \wedge (d\eta)^{n-1} \neq 0$ everywhere on $\overline{M}$, where the exponent denotes the $(n-1)$-th exterior power. We call such η a *contact form* of $\overline{M}$. It is well known that given a contact form η, there exists a unique vector field ξ, which is called the *characteristic vector field*, satisfying $\eta(\xi) = 1$ and $d\eta(\xi, \overline{X}) = 0$ for any vector field $\overline{X}$ on $\overline{M}$. A Riemannian metric $\bar{g}$ is an associated metric to a contact form η if there exists a $(1,1)$-tensor field ϕ satisfying

$$\eta(\overline{X}) = \bar{g}(\overline{X}, \xi), \quad d\eta(\overline{X}, \overline{Y}) = \bar{g}(\overline{X}, \phi\overline{Y}), \quad \phi^2\overline{X} = -\overline{X} + \eta(\overline{X})\xi \tag{2.1}$$

where $\overline{X}$ and $\overline{Y}$ are vector fields on $\overline{M}$. From (2.1) it follows that

$$\phi\xi = 0, \quad \eta \circ \phi = 0, \quad \bar{g}(\phi\overline{X}, \phi\overline{Y}) = \bar{g}(\overline{X}, \overline{Y}) - \eta(\overline{X})\eta(\overline{Y}).$$

A Riemannian manifold $\overline{M}$ equipped with structure tensors $(\eta, \bar{g}, \phi, \xi)$ satisfying (2.1) is said to be a *contact metric manifold*. We assume that a contact metric manifold $\overline{M} = (\overline{M}, \eta, \bar{g}, \phi, \xi)$ is always oriented by the $(2n-1)$-form $\eta \wedge (d\eta)^{n-1}$. We denote by dV the volume form of $\overline{M}$ with respect to the metric $\bar{g}$. Then we may easily observe that $dV = C\eta \wedge (d\eta)^{n-1}$, where $C = \frac{1}{2^{n-1}(n-1)!}$. We now review some elementary facts in a contact metric manifold. First, for the characteristic vector field ξ, $L_\xi\eta = 0$ follows from $\eta(\xi) = 1$, $d\eta(\overline{X}, \overline{Y}) = \bar{g}(\overline{X}, \phi\overline{Y})$ and $d\eta(\xi, \overline{X}) = 0$. Here L denotes Lie derivation. Next, since $d \circ L_\xi = L_\xi \circ d$, by taking account of $L_\xi\eta = 0$, we have

$$\begin{aligned} L_\xi dV &= CL_\xi(\eta \wedge (d\eta)^{n-1}) \\ &= C(L_\xi\eta) \wedge (d\eta)^{n-1} + C\eta \wedge (L_\xi d\eta) \wedge d\eta \wedge \cdots \wedge d\eta \\ &\quad + \cdots + C\eta \wedge d\eta \wedge \cdots \wedge (L_\xi d\eta) \\ &= C\eta \wedge d(L_\xi\eta) \wedge d\eta \wedge \cdots \wedge d\eta + \cdots + C\eta \wedge d\eta \wedge \cdots \wedge d(L_\xi\eta) \\ &= 0. \end{aligned} \tag{2.2}$$

Since $L_\xi dV = (\mathrm{div}\xi)dV$, by the definition of the divergence $\mathrm{div}\xi$ with respect to dV and by (2.2), we have

$$\mathrm{div}\xi = 0 \quad (\text{i.e., } \overline{\nabla}_i \xi^i = 0). \tag{2.3}$$

Since $\overline{\nabla}_{\overline{X}}\xi$ is orthogonal to ξ, we have immediately

$$(\overline{\nabla}_{\overline{X}}\eta)\xi = 0 \tag{2.4}$$

for any vector field $\overline{X}$ on $\overline{M}$.

Let (M,g) be an n-dimensional Riemannian manifold and ∇ the associated Levi Civita connection. Its Riemann curvature tensor R is defined by $R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$ for all vector fields X, Y, and Z on M. The tangent bundle of (M,g) is denoted by TM and consists of pairs (p,u), where p is a point in M and u a tangent vector to M at P. The mapping $\pi : TM \to M$, $\pi(p,u) = p$ is the natural projection from TM onto M. For a vector field X on M, its *vertical lift* X^v on TM is the vector field defined by $X^v\omega = \omega(X)\circ\pi$, where ω is a 1-form on M. For a Levi Civita connection ∇ on M, the *horizontal lift* X^h of X is defined by $X^h\omega = \nabla_X\omega$. The tangent bundle TM can be endowed in a natural way with a Riemannian metric $\tilde{g}$, the so-called *Sasaki metric*, depending only on the Riemannian metric g on M. It is determined by

$$\tilde{g}(X^h,Y^h) = \tilde{g}(X^v,Y^v) = g(X,Y)\circ\pi, \quad \tilde{g}(X^h,Y^v) = 0$$

for all vector fields X and Y on M. Also, TM admits an almost complex structure tensor J defined by $JX^h = X^v$ and $JX^v = -X^h$. Then g is a Hermitian metric for the almost complex structure J. We note that J is integrable if and only if (M,g) is locally flat ([7]).

The unit tangent bundle $\bar{\pi} : T_1M \to M$ is a hypersurface of TM given by $g_p(u,u) = 1$. Note that $\bar{\pi} = \pi\circ i$, where i is the immersion. A unit normal vector $N = u^v$ to T_1M is given by the vertical lift of u for (p,u). The horizontal lift of a vector is tangent to T_1M, but the vertical lift of vector is not tangent to T_1M in general. So, we define the *tangential lift* of X to $(p,u)\in T_1M$ by

$$X^t_{(p,u)} = (X - g(X,u)u)^v.$$

Clearly, the tangent space $T_{(p,u)}T_1M$ is spanned by vectors of the form X^h and X^t, where $X \in T_pM$.

We now define the canonical contact metric structure of the unit tangent bundle T_1M of a Riemannian manifold (M,g). The metric g' on T_1M is induced from the Sasaki metric $\tilde{g}$ on TM. Using the almost complex structure J on TM, we define a unit vector field ξ', a 1-form η' and a (1,1)-tensor field ϕ' on T_1M by

$$\xi' = -JN, \quad \phi' = J - \eta'\otimes N.$$

Since $g'(\overline{X},\phi'\overline{Y}) = 2d\eta'(\overline{X},\overline{Y})$, (η',g',ϕ',ξ') is not a contact metric structure. If we rescale by

$$\xi = 2\xi', \quad \eta = \frac{1}{2}\eta', \quad \phi = \phi', \quad \bar{g} = \frac{1}{4}g',$$

we get the canonical contact metric structure $(\eta, \bar{g}, \phi, \xi)$. These tensors are given by

$$\begin{aligned}\xi &= 2u^h,\\ \phi X^t &= -X^h + \frac{1}{2}g(X,u)\xi, \quad \phi X^h = X^t,\\ \eta(X^t) &= 0, \quad \eta(X^h) = \frac{1}{2}g(X,u),\\ \bar{g}(X^t,Y^t) &= \frac{1}{4}(g(X,Y) - g(X,u)g(Y,u)),\\ \bar{g}(X^t,Y^h) &= 0,\\ \bar{g}(X^h,Y^h) &= \frac{1}{4}g(X,Y),\end{aligned} \tag{2.5}$$

where X and Y are vector fields on M. From now on, we consider $T_1M = (T_1M, \eta, \bar{g}, \phi, \xi)$ with the canonical contact metric structure.

The Levi Civita connection $\overline{\nabla}$ of T_1M is described by

$$\begin{aligned}\overline{\nabla}_{X^t}Y^t &= -g(Y,u)X^t,\\ \overline{\nabla}_{X^t}Y^h &= \frac{1}{2}(R(u,X)Y)^h,\\ \overline{\nabla}_{X^h}Y^t &= (\nabla_X Y)^t + \frac{1}{2}(R(u,Y)X)^h,\\ \overline{\nabla}_{X^h}Y^h &= (\nabla_X Y)^h - \frac{1}{2}(R(X,Y)u)^t\end{aligned} \tag{2.6}$$

for all vector fields X and Y on M.

Also the Riemann curvature tensor $\overline{R}$ of T_1M is given by

$$\begin{aligned}\overline{R}(X^t,Y^t)Z^t &= -(g(X,Z) - g(X,u)g(Z,u))Y^t + (g(Y,Z) - g(Y,u)g(Z,u))X^t,\\ \overline{R}(X^t,Y^t)Z^h &= \{R(X - g(X,u)u, Y - g(Y,u)u)Z\}^h + \frac{1}{4}\{[R(u,X),R(u,Y)]Z\}^h,\\ \overline{R}(X^h,Y^t)Z^t &= -\frac{1}{2}\{R(Y - g(Y,u)u, Z - g(Z,u)u)X\}^h - \frac{1}{4}\{R(u,Y)R(u,Z)X\}^h,\\ \overline{R}(X^h,Y^t)Z^h &= \frac{1}{2}\{R(X,Z)(Y - g(Y,u)u)\}^t - \frac{1}{4}\{R(X,R(u,Y)Z)u\}^t\\ &\quad + \frac{1}{2}\{(\nabla_X R)(u,Y)Z\}^h,\\ \overline{R}(X^h,Y^h)Z^t &= \{R(X,Y)(Z - g(Z,u)u)\}^t + \frac{1}{4}\{R(Y,R(u,Z)X)u - R(X,R(u,Z)Y)u\}^t\\ &\quad + \frac{1}{2}\{(\nabla_X R)(u,Z)Y - (\nabla_Y R)(u,Z)X\}^h,\\ \overline{R}(X^h,Y^h)Z^h &= (R(X,Y)Z)^h + \frac{1}{2}\{R(u,R(X,Y)u)Z\}^h\\ &\quad - \frac{1}{4}\{R(u,R(Y,Z)u)X - R(u,R(X,Z)u)Y\}^h + \frac{1}{2}\{(\nabla_Z R)(X,Y)u\}^t\end{aligned} \tag{2.7}$$

for all vector fields X, Y, and Z on M.

Next, to calculate the Ricci tensor $\bar{\rho}$ of T_1M at the point $(p,u)\in T_1M$, let $e_1,\dots,e_n=u$ be an orthonormal basis of T_pM. Then $2e_1^t,\dots,2e_{n-1}^t$, $2e_1^h,\dots,2e_n^h=\xi$, is an orthonormal basis for $T_{(p,u)}T_1M$ and $\bar{\rho}$ is given by

$$
\begin{aligned}
\bar{\rho}(X^t,Y^t) &= (n-2)(g(X,Y)-g(X,u)g(Y,u))+\frac{1}{4}\sum_{i=1}^{n} g(R(u,X)e_i,R(u,Y)e_i),\\
\bar{\rho}(X^t,Y^h) &= \frac{1}{2}((\nabla_u\rho)(X,Y)-(\nabla_X\rho)(u,Y)),\\
\bar{\rho}(X^h,Y^h) &= \rho(X,Y)-\frac{1}{2}\sum_{i=1}^{n} g(R(u,e_i)X,R(u,e_i)Y),
\end{aligned}
\tag{2.8}
$$

where ρ denotes the Ricci curvature tensor of M. From this, the scalar curvature $\bar{\tau}$ is given by

$$
\frac{\bar{\tau}}{4}=\tau+(n-1)(n-2)-\frac{1}{4}\sum_{i,j=1}^{n} g(R(u,e_i)e_j,R(u,e_i)e_j),
\tag{2.9}
$$

where τ is the scalar curvature of M.

3. Unit tangent bundle with η-Einstein structure

We shall introduce the definition of η-Einstein manifold.

Definition 1. If the Ricci tensor $\bar{\rho}$ of a contact metric manifold $(\overline{M},\eta,\bar{g},\phi,\xi)$ is of the form

$$
\bar{\rho}(\overline{X},\overline{Y})=\alpha\,\bar{g}(\overline{X},\overline{Y})+\beta\,\eta(\overline{X})\eta(\overline{Y})
$$

for smooth functions α and β, then $\overline{M}$ is called an η-Einstein manifold.

Now, let $M=(M,g)$ be a Riemannian manifold and $(T_1M,\eta,\bar{g},\phi,\xi)$ be the unit tangent bundle of (M,g) equipped with the canonical contact metric structure $(\eta,\bar{g},\phi,\xi)$ defined as in Section 2. Take the ϕ-basis $\{\bar{e}_i,\bar{e}_{i^*}=\phi\bar{e}_i,\xi=\bar{e}_*\}$ on T_1M. Then the Ricci tensor $\bar{\rho}$ with respect to the ϕ-basis should be

$$
\bar{\rho}=\begin{pmatrix}
\bar{\rho}(\bar{e}_i,\bar{e}_j) & \bar{\rho}(\bar{e}_i,\bar{e}_{j^*}) & \bar{\rho}(\bar{e}_i,\bar{e}_*)\\
\bar{\rho}(\bar{e}_{i^*},\bar{e}_j) & \bar{\rho}(\bar{e}_{i^*},\bar{e}_{j^*}) & \bar{\rho}(\bar{e}_{i^*},\bar{e}_*)\\
\bar{\rho}(\bar{e}_*,\bar{e}_j) & \bar{\rho}(\bar{e}_*,\bar{e}_{j^*}) & \bar{\rho}(\bar{e}_*,\bar{e}_*
\end{pmatrix}.
\tag{3.1}
$$

In particular, if T_1M is η-Einstein, by the definition, the Ricci tensor $\bar{\rho}$ is given by

$$
\bar{\rho}=\begin{pmatrix}
\alpha & o & \cdots & 0 & 0\\
0 & \alpha & \cdots & 0 & 0\\
\vdots & \vdots & & \vdots & \vdots\\
0 & 0 & \cdots & \alpha & 0\\
0 & 0 & \cdots & 0 & \alpha+\beta
\end{pmatrix}
\tag{3.2}
$$

for some two smooth functions α and β on T_1M. From (2.8), we have the following theorem.

Theorem 4. *Let M be an n-dimensional Riemannian manifold. Then T_1M is η-Einstein if and only if*

$$\sum_{i=1}^{n} g(R(u,X)e_i, R(u,Y)e_i) = (\alpha - 4n + 8)(g(X,Y) - g(X,u)g(Y,u)), \tag{3.3}$$

$$(\nabla_u \rho)(X,Y) = (\nabla_X \rho)(u,Y), \tag{3.4}$$

$$\sum_{i=1}^{n} g(R(u,e_i)X, R(u,e_i)Y) = 2\rho(X,Y) - \frac{1}{2}\alpha\, g(X,Y) - \frac{1}{2}\beta\, g(X,u)g(Y,u). \tag{3.5}$$

Proof of Theorem 1. Let $T_1M = (T_1M, \eta, \bar{g}, \phi, \xi)$ be the unit tangent bundle equipped with the canonical contact metric structure $(\eta, \bar{g}, \phi, \xi)$ and assume that T_1M is an η-Einstein manifold. Then, by the definition, the Ricci tensor $\bar{\rho}$ of T_1M takes of the following form:

$$\bar{\rho} = \alpha \bar{g} + \beta \eta \otimes \eta \tag{3.6}$$

for some smooth functions α and β on T_1M.

For a while, we adopt the traditional convention for the notations in the classical tensor analysis. In the local coordinate neighborhood, from (3.6), we get

$$\bar{\rho}_{ij} = \alpha \bar{g}_{ij} + \beta \eta_i \eta_j. \tag{3.7}$$

Operating $\overline{\nabla}^i = \bar{g}^{ia}\overline{\nabla}_a$ on both sides of (3.7), we get

$$\begin{aligned}\overline{\nabla}^i \bar{\rho}_{ij} &= (\overline{\nabla}^i \alpha)\bar{g}_{ij} + (\overline{\nabla}^i \beta)\eta_i\eta_j + \beta(\overline{\nabla}^i \eta_i)\eta_j + \beta\eta_i(\overline{\nabla}^i \eta_j)\\ &= \overline{\nabla}_j \alpha + (\overline{\nabla}_i \beta)\xi^i \eta_j + \beta(\mathrm{div}\xi)\eta_j + \beta\xi^i \overline{\nabla}_i \eta_j.\end{aligned} \tag{3.8}$$

Transvecting ξ^j with (3.8), we have

$$\xi^j \overline{\nabla}^i \bar{\rho}_{ij} = \xi\alpha + \xi\beta + \beta(\mathrm{div}\xi) + \beta(\overline{\nabla}_\xi \eta)\xi. \tag{3.9}$$

Here, taking account of the second Bianchi identity, we get

$$\overline{\nabla}^i \bar{\rho}_{ij} = \frac{1}{2}\overline{\nabla}_j \bar{\tau}$$

and hence the left-hand side of (3.9) implies $\frac{1}{2}\xi\bar{\tau}$. Thus, from (2.3), (2.4), and (3.9) we have

$$\xi\bar{\tau} = 2\xi\alpha + 2\xi\beta. \tag{3.10}$$

On one hand, by (3.7), we get

$$\bar{\tau} = (2n-1)\alpha + \beta.$$

Thus, we have also

$$\xi\bar{\tau} = (2n-1)\xi\alpha + \xi\beta. \tag{3.11}$$

Then from (3.10) and (3.11), we have

$$(2n-3)\xi\alpha - \xi\beta = 0. \tag{3.12}$$

Next, let $\overline{X} = (X^j)$ be a vector field on T_1M with $\eta(\overline{X}) = 0$. Transvecting X^j with (3.8), we have also

$$X^j\overline{\nabla}^i\bar{\rho}_{ij} = \overline{X}\alpha + \beta(\overline{\nabla}_\xi\eta)(\overline{X})$$

and hence

$$\frac{1}{2}\overline{X}\bar{\tau} = \overline{X}\alpha + \beta(\overline{\nabla}_\xi\eta)(\overline{X}). \tag{3.13}$$

Here, we get

$$\begin{aligned}(\overline{\nabla}_\xi\eta)(\overline{X}) &= -\eta(\overline{\nabla}_\xi\overline{X})\\ &= -\eta(\overline{\nabla}_{\overline{X}}\xi + [\xi,\overline{X}])\\ &= -\eta([\xi,\overline{X}]).\end{aligned} \tag{3.14}$$

On one hand, we get

$$\begin{aligned}-\eta([\xi,\overline{X}]) &= \xi(\eta(\overline{X})) - \overline{X}(\eta(\xi)) - \eta([\xi,\overline{X}])\\ &= d\eta(\xi,\overline{X})\\ &= \bar{g}(\xi,\phi\overline{X})\\ &= 0.\end{aligned} \tag{3.15}$$

Thus from (3.13)–(3.15), we have

$$\overline{X}\bar{\tau} = 2\overline{X}\alpha \tag{3.16}$$

for vector field $\overline{X}$ with $\eta(\overline{X}) = 0$. Since $\bar{\tau} = (2n-1)\alpha + \beta$ holds on T_1M, we have also

$$\overline{X}\bar{\tau} = (2n-1)\overline{X}\alpha + \overline{X}\beta. \tag{3.17}$$

Thus, by (3.16) and (3.17), we have

$$(2n-3)\overline{X}\alpha + \overline{X}\beta = 0 \tag{3.18}$$

for vector field $\overline{X}$ with $\eta(\overline{X}) = 0$.

From now on, we state some fundamental properties of the η-Einstein contact metric structure $(\eta, \bar{g}, \phi, \xi)$ on T_1M, by making use of the facts in the above. First of all, by (3.4), we see that the scalar curvature τ of the base manifold (M, g) ($\dim M \geqslant 2$) is constant.

Now setting $X = Y = e_j$ in (3.3) and (3.5) and taking sum for $j = 1, \ldots, n$, we obtain

$$\sum_{i,j=1}^{n} g(R(u,e_j)e_i, R(u,e_j)e_i) = (\alpha - 4n + 8)(n-1), \tag{3.19}$$

$$\sum_{i,j=1}^{n} g(R(u,e_i)e_j, R(u,e_i)e_j) = 2\tau - \frac{1}{2}\,n\alpha - \frac{1}{2}\beta. \tag{3.20}$$

From (3.19) and (3.20), we have

$$(3n-2)\alpha + \beta = 4\tau + 8(n-1)(n-2). \tag{3.21}$$

Since τ is constant and $\xi = 2u^h$, we have

$$(3n-2)u^h\alpha + u^h\beta = 0. \tag{3.22}$$

And (3.12) can be rewritten as follows:

$$(2n-3)u^h\alpha - u^h\beta = 0. \tag{3.23}$$

From (3.22) and (3.23), we have

$$u^h\alpha = 0 \quad \text{and } u^h\beta = 0. \tag{3.24}$$

Operating $X^t(X \in T_pM) \in T_{(p,u)}T_1M$ on the both sides of (3.21), we have

$$(3n-2)X^t\alpha + X^t\beta = 0. \tag{3.25}$$

Since X^t is orthogonal to ξ (i.e., $\eta(X^t) = 0$), from (3.18), we have

$$(2n-3)X^t\alpha + X^t\beta = 0. \tag{3.26}$$

Thus, from (3.25) and (3.26), we have

$$X^t\alpha = 0 \quad \text{and } X^t\beta = 0 \text{ at } (p,u). \tag{3.27}$$

Similarly, operating $X^h(X \in T_pM) \in T_{(p,u)}T_1M$ on the both sides of (3.21) for vector field X on M such that $g(X,u) = 0$, we have

$$X^h\alpha = 0 \quad \text{and } X^h\beta = 0 \text{ at } (p,u). \tag{3.28}$$

Summing up (3.24), (3.27) and (3.28), we see that the smooth functions α and β are constants. □

By Theorem 1, we immediately obtain that

Corollary 5. *T_1M with η-Einstein structure has constant scalar curvature $\bar{\tau}$.*

Proof of Theorem 2. For T_1M with constant scalar curvature it holds

$$\sum_{i,j=1}^{n} g(R(u,e_j)e_i, R(u,e_j)e_i) = \frac{|R|^2}{n}, \tag{3.29}$$

where $|R|^2 = \sum_{i,j,k=1}^{n} g(R(e_i,e_j)e_k, R(e_i,e_j)e_k)$ ([4]). From (3.19), (3.20), and (3.29), we have

$$\alpha = \frac{|R|^2}{n(n-1)} + 4(n-2), \tag{3.30}$$

$$\beta = 4\tau - 4n(n-2) - \frac{3n-2}{n(n-1)}|R|^2. \tag{3.31}$$

Next, we integrate (3.5) with $X = Y = u$ over $S^{n-1}(1)$ in T_pM. Then using the formula in [6], we have

$$\frac{1}{n(n+2)}\left(|\rho|^2 + \frac{3}{2}|R|^2\right) = \frac{2\tau}{n} - \frac{1}{2}\alpha - \frac{1}{2}\beta. \tag{3.32}$$

Eliminating α and β from (3.30)–(3.32), we obtain the equation:

$$2|\rho|^2 - 3(n+1)|R|^2 = -4(n-1)(n+2)\tau + 4n(n-1)(n-2)(n+2). \tag{3.33}$$

In proof of Theorem 1, we know that α, β and τ are constant. Since α is constant, from (3.30), we see that $|R|^2$ is constant. Thus, by (3.32), we see also that $|\rho|^2$ is constant. Therefore we have Theorem 2. □

4. Special cases

(I) 2-dimensional case. It is well-known that $R(X,Y)Z = \kappa(g(Y,Z)X - g(X,Z)Y)$ always holds. So, we have $|R|^2 = 4\kappa^2$, $|\rho|^2 = 2\kappa^2$, $= 2\kappa$, where κ is the Gaussian curvature. From (3.33), we see that M has Gaussian curvature $\kappa = 0$ or $\kappa = 1$.

(II) 3-dimensional case. It is well-known that the curvature tensor R of 3-dimensional Riemannian manifold (M,g) is of the following form.

$$\begin{aligned} R(X,Y,Z,W) &= g(R(X,Y)Z,W) \\ &= \{g(X,W)\rho(Y,Z) + g(Y,Z)\rho(X,W) \\ &\quad - g(X,Z)\rho(Y,W) - g(Y,W)\rho(X,Z)\} \\ &\quad + \frac{\tau}{2}\{g(X,Z)g(Y,W) - g(Y,Z)g(X,W)\} \end{aligned} \tag{4.1}$$

for all vector fields X, Y, Z, W on M. From (4.1), by direct calculation, we get

$$|R|^2 = 4|\rho|^2 - \tau^2. \tag{4.2}$$

By (3.33) and (4.2), we have

$$23|\rho|^2 - 6\tau^2 - 20\tau + 60 = 0,$$

and thus

$$23\left|\rho - \frac{\tau}{3}g\right|^2 + \frac{5}{3}(\tau - 6)^2 = 0. \tag{4.3}$$

From (4.3), we have $\rho = \frac{\tau}{3}g$ and $\tau = 6$ and hence

$$\rho = 2g. \tag{4.4}$$

Thus by (4.1) and (4.4), we have

$$R(X,Y,Z,W) = g(X,W)g(Y,Z) - g(X,Z)g(Y,W)$$

and hence (M,g) is a space of constant sectional curvature 1. The above result has been proved in [5]. We may note that our proof is much simpler than their proof.

(III) Conformally flat case. By the similar arguments in [5], we can also have the following.

Theorem 6. *Let $M = (M,g)$ be an n-dimensional conformally flat manifold $(n \geqslant 4)$. Then $(T_1M, \eta, \bar{g}, \phi, \xi)$ is η-Einstein if and only if (M,g) is a space of constant sectional curvature 1 or $n-2$.*

(IV) Einstein case. Let $M = (M,g)$ be an n-dimensional Einstein manifold $(n \geqslant 3)$. Then we have

$$\left|R + \frac{\tau}{2n(n-1)} g \owedge g\right|^2 = |R|^2 - \frac{2\tau^2}{n(n-1)}, \tag{4.5}$$

where $(h \owedge k)(X,Y,Z,W) = h(X,Z)k(Y,W) + h(Y,W)k(X,Z) - h(X,W)k(Y,Z) - h(Y,Z)k(X,W)$ for any (0,2)-tensors h and k. By (3.33) and (4.5), we have

$$\begin{aligned} &\frac{2\tau^2}{n} - 3(n+1)\left\{ \left| R + \frac{\tau}{2n(n-1)} g \owedge g \right|^2 + \frac{2\tau^2}{n(n-1)} \right\} \\ &\qquad = -4(n-1)(n+2)\tau + 4n(n-1)(n-2)(n+2) \end{aligned}$$

and hence

$$\begin{aligned} &-3(n+1)\left| R + \frac{\tau}{2n(n-1)} g \owedge g \right|^2 \\ &\quad = \frac{4(n+2)}{n(n-1)} \{\tau^2 - n(n-1)^2\tau + n^2(n-1)^2(n-2)\} \\ &\quad = \frac{4(n+2)}{n(n-1)} (\tau - n(n-1))(\tau - n(n-1)(n-2)). \end{aligned} \tag{4.6}$$

Then from (4.6), we have

$$n(n-1) \leqslant \tau \leqslant n(n-1)(n-2), \quad n \geqslant 3. \tag{4.7}$$

By Theorem 4, we see that (M,g) is super-Einstein by virtue of (3.5). Since the scalar curvature of T_1M is constant as shown by Theorem 2, this also follows from the result of Boeckx and Vanhecke ([4], Proposition 3.6). Thus we have

Theorem 7. *Let (M,g) be an n-dimensional Einstein manifold and $(T_1M, \eta, \bar{g}, \phi, \xi)$ be the unit tangent bundle of M equipped with the canonical contact metric structure. If T_1M is η-Einstein, then M is super-Einstein and the scalar curvature τ satisfies the above inequality* (4.7).

In the remainder of this section, we shall consider the case that (M,g) is a 4-dimensional Einstein manifold.

Proof of Theorem 3. We may choose an orthonormal basis $\{e_i\}$ (known as the Singer-Thorpe basis) at each point $p \in M$ such that

$$\begin{cases} R_{1212} = R_{3434} = a, \quad R_{1313} = R_{2424} = b, \quad R_{1414} = R_{2323} = c, \\ R_{1234} = d, \quad R_{1342} = e, \quad R_{1423} = f, \\ R_{ijkl} = 0 \quad \text{whenever just three of the indices } i,j,k,l \text{ are distinct([10]).} \end{cases} \tag{4.8}$$

Note that $d+e+f=0$ by the first Bianchi identity and

$$a+b+c = -\frac{\tau}{4}. \tag{4.9}$$

Further, by the direct calculation, we have

$$\begin{aligned} |R|^2 &= 8(a^2+b^2+c^2+d^2+e^2+f^2), \\ |\rho|^2 &= 4(a+b+c)^2. \end{aligned} \tag{4.10}$$

From Theorem 7, since M is super-Einstein, we have ([8], [9])

$$\pm d = a + \frac{\tau}{12}, \quad \pm e = b + \frac{\tau}{12}, \quad \pm f = c + \frac{\tau}{12}. \tag{4.11}$$

From (3.3), taking account of (4.8), we have easily

$$2(a^2+d^2)=\alpha-8, \tag{4.12}$$

$$2(b^2+e^2)=\alpha-8, \tag{4.13}$$

$$2(c^2+f^2)=\alpha-8. \tag{4.14}$$

Thus from (4.12) and (4.13), taking account of (4.11), we have

$$(a-b)\left(a+b+\frac{\tau}{12}\right)=0. \tag{4.15}$$

Similarly, we have

$$(b-c)\left(b+c+\frac{\tau}{12}\right)=0, \tag{4.16}$$

$$(c-a)\left(c+a+\frac{\tau}{12}\right)=0. \tag{4.17}$$

We first suppose that $a\neq b$, $b\neq c$, $c\neq a$. Then by (4.15)–(4.17), we get

$$a+b+\frac{\tau}{12}=0, \quad b+c+\frac{\tau}{12}=0, \quad c+a+\frac{\tau}{12}=0.$$

Thus we have $a=b=c=-\frac{\tau}{24}$. However this is a contradiction.

Next, we suppose that $a\neq b$, $b\neq c$, $c=a$ (i.e., $a=c$, $a\neq b$). Then by (4.15)–(4.17), we have

$$a+b+\frac{\tau}{12}=0, \quad b+c+\frac{\tau}{12}=0. \tag{4.18}$$

By (4.9) and the hypothesis $a=c$, we have

$$2a+b=-\frac{\tau}{4}. \tag{4.19}$$

Thus by (4.18) and (4.19), we have

$$a=c=-\frac{\tau}{6}, \quad b=\frac{\tau}{12}. \tag{4.20}$$

Thus by (4.11) and (4.20), we have

$$\pm d=-\frac{\tau}{12}, \quad \pm e=\frac{\tau}{6}, \quad \pm f=-\frac{\tau}{12}. \tag{4.21}$$

Thus from (4.10), (4.20), and (4.21), we have

$$|R|^2=\frac{5}{6}\tau^2, \quad |\rho|^2=\frac{\tau^2}{4}. \tag{4.22}$$

Then by (3.33) and (4.22), we obtain

$$\tau^2-6\tau+48=0. \tag{4.23}$$

However, this quadratic Equation (4.23) does not admit a real solution. This is also a contradiction. By the similar way, we can also deduce a contradiction in the cases

$b = c \neq a$ and $a = b \neq c$. Thus, it must follow that $a = b = c$ and hence by (4.9) and (4.11), we have

$$a = b = c = -\frac{\tau}{12}, \quad d = e = f = 0.$$

Therefore, by (4.8), (M, g) is a space of constant sectional curvature $\frac{\tau}{12}$. Then we have

$$|R|^2 = \frac{\tau^2}{6}, \quad |\rho|^2 = \frac{\tau^2}{4}.$$

Thus, by (3.33), we have

$$(\tau - 12)(\tau - 24) = 0. \tag{4.24}$$

Therefore we have Theorem 3. □

Acknowledgements. Research of Y. D. Chai was partially supported by KOSEF R01-2004-000-10183-0(2006), research of S. H. Chun was supported by the Korean Research Foundation Grant funded by the Korean Government (MOEHRD) (KRF-2006-351-C00003), and research of J. H. Park was partially supported by the Korea Research Foundation Grant funded by the Korean Government (MOEHRD) KRF-2007-531-C00008.

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